RESEARCH INTO ADVANCED CONCEPTS

OF MICROWAVE POWER AMPLIFICATION

AND

GENERATION UTILIZING LINEAR BEAM DEVICES

P. R. McIsaac

School of Electrical Engineering

Cornell University

Semiannual Status Report

5-31-68

Research Grant NGL 33-010-047 No. 2

National Aeronautics and Space Administration

Electronics Research Center

Cambridge, Massachusetts

ABSTRACT

This is an interim report which summarizes work during the past six months on a theoretical study of some aspects of the interaction between a drifting stream of electrons with transverse cyclotron motions and an electromagnetic field. Particular emphasis is given to the possible generation and amplification of millimeter waves. The report includes brief discussions of two aspects of the beam waves of a spiraling filamentary electron beam. The first is a preliminary coupled mode analysis of the interaction between the beam waves and the TE waves of a circuit, and the second is the determination of the power associated with the beam waves when they are excited by TEM circuit waves.

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I. INTRODUCTION

The objective of this research program is to explore theoretically some aspects of the interaction between a drifting stream of electrons having transverse cyclotron motions and an electromagnetic field; particular emphasis being given to the possible generation and amplification of millimeter waves. Because of the interest in possible applications to millimeter wavelengths, this study concentrates on electron stream - electromagnetic field interactions which involve a uniform, or fast-wave, circuit structure.

This interim report summarizes work on two aspects of the interaction between spiraling electron beams and electromagnetic fields. Section II extends the previous coupledmode analysis[1] of the interaction between a spiraling filamentary electron beam and TEM circuit waves to the interaction with TE circuit waves. Section III is concerned with the development of a procedure for determining the power associated with the beam waves of a spiraling filamentary electron beam. Using this procedure the power associated with several of the beam waves is determined when they are excited by various TEM circuit waves.

II. SPIRALING FILAMENTARY ELECTRON BEAM INTERACTION WITH A TE WAVE

A. Basic Equations

The previous semiannual status report[1] presented some aspects of the interaction between a spiraling filamentary electron beam and the TEM waves of a uniform circuit. That analysis is extended here to consider the interaction with the TE waves of a uniform circuit. Again, a coupled mode, small signal, analysis is employed, and relativistic effects are included. The model for the spiraling filamentary electron beam, together with the d-c and first order a-c electron beam equations were presented in the previous report and will not be repeated here.

The interaction circuit considered is a square waveguide of edge a. Of the infinite set of modes supported by this waveguide, attention is restricted to combinations of the dominant TE_{10} and TE_{01} modes which provide positive and negative circularly polarized waves at the center of the waveguide. The axis of the spiraling filamentary electron beam is assumed to coincide with the waveguide axis.

There are two major differences in this case of TE wave interaction compared with the TEM wave interaction considered previously. First, an axial a-c magnetic field is now present which contributes to the transverse acceleration of the electrons. Second, the fields of the TE waves are not uniform over the cross section of the waveguide (the TEM wave considered previously was uniform in the transverse plane). Both of these effects are included in the analysis.

The six beam waves, P'_+, P'_-, V', Q'_+, Q'_-, and W' are defined essentially as before (Equations (17a) - (17f) of reference 1). The normalizing factor M for these beam waves is changed slightly to conform with the waveguide circuit now considered.

$$M = \frac{\sqrt{\pi}}{8} \sqrt{\frac{(1 + \eta^2 - \sigma^2)}{\eta^3} \quad \frac{mI_o}{e}} \tag{1}$$

The circularly polarized circuit wave amplitudes are defined as

$$F'_{\pm} = \frac{a}{4\sqrt{2Z}} \left(E'_{\pm} \mp jH'_{\pm}\right), \tag{2a}$$

$$G'_{\pm} = \frac{a}{4\sqrt{2Z}} \left(E'_{\pm} \pm jH'_{\pm}\right), \tag{2b}$$

where $F'_{\pm}$ represent positive and negative circularly polarized waves traveling in the +z direction, and $G'_{\pm}$ represent similar waves traveling in the -z direction. Here Z is the impedance of the TE_{10} mode of the square waveguide. The circularly polarized electric and magnetic field amplitudes which appear in these equations are based on the electric and magnetic fields at the center of the waveguide.

$$E'_{\pm} = (E_x(0,0) \pm jE_y(0,0))e^{\mp j\psi} \tag{3a}$$

$$H'_{\pm} = (H_x(0,0) \pm jH_y(0,0))e^{\mp j\psi} \tag{3b}$$

$$\psi = \eta \beta_c z + \phi \tag{4}$$

The axial magnetic field H_z can be expressed in terms of the transverse components of the electric field using the usual expressions for the fields of a rectangular waveguide. Hence the axial magnetic field amplitude can be expressed in terms of the circuit wave amplitudes, $F'_{\pm}$ and $G'_{\pm}$.

The coupled mode equations for the circuit wave amplitudes are obtained from Maxwell's equations by averaging over the cross section of the waveguide. The electron beam current enters as a driving term, but the precise location of the electrons within the cross section does not affect these circuit equations.

In the equations of motion, however, the electric and magnetic fields which cause the forces acting on the electrons must be evaluated at the electrons' location. That is, the a-c electric and magnetic fields must be evaluated along the d-c spiral trajectory of the electron beam, although the wave amplitudes are expressed in terms of the fields at the center of the waveguide. Recalling that the d-c components of the transverse position of the electrons are $x_o = r_o \cos\psi$, $y_o = r_o \sin\psi$, where r_o is the radius of the electron beam spiral, the electric and magnetic fields at the beam location can be expressed in terms of series of Bessel functions of the first kind.

$$E_{\pm}(x_o,y_o) = E_x(0,0)\cos(\pi y_o/a) \pm jE_y(0,0)\cos(\pi x_o/a)$$
$$= \left[J_o(\pi r_o/a) + 2\sum_{n=1}^{\infty} J_{4n}(\pi r_o/a)\cos(4n\psi)\right]E_{\pm}$$
$$+ 2\sum_{n=1}^{\infty} J_{4n-2}(\pi r_o/a)\cos((4n-2)\psi)E_{\mp} \tag{5a}$$

$$H_{\pm}(x_o,y_o) = H_x(0,0)\cos(\pi x_o/a) \pm jH_y(0,0)\cos(\pi y_o/a)$$
$$= \left[J_o(\pi r_o/a)+2\sum_{n=1}^{\infty} J_{4n}(\pi r_o/a)\cos(4n\psi)\right]H_{\pm}$$
$$- 2\sum_{n=1}^{\infty} J_{4n-2}(\pi r_o/a)\cos((4n-2)\psi)\ H_{\mp} \tag{5b}$$

$$H_z(x_o, y_o) = \frac{j\pi}{\omega\mu a}\left[E_x(0,0)\sin(\pi y_o/a) - E_y(0,0)\sin(\pi x_o/a)\right]$$

$$= \frac{\pi}{\omega\mu a}\sum_{n=1}^{\infty} (-1)^n J_{2n-1}(\pi r_o/a)\Big[E_+ e^{j(-1)^n(2n-1)\psi}$$

$$-E_- e^{-j(-1)^n(2n-1)\psi}\Big] \qquad (5c)$$

It is these field quantities which are to be used in the equations of motion for the electron beam.

The ten coupled mode equations for this system of six electron beam waves and four circuit waves are obtained, as before, from the equations of motion for the electron beam and Maxwell's equations.

$$\left(\frac{\partial}{\partial z} + j\beta_e\right)P_+' + j\,\frac{2\lambda}{1-\lambda}\,\eta\,\beta_c\,P_-' + K\left\{\frac{1}{1-\lambda}\,(A_1 + A_2) - \frac{\sigma\lambda}{\eta^2(1+\lambda)}\,\frac{\eta\beta_c}{\beta_e}\left[\frac{2\sigma}{1+\lambda}\,(A_1 - A_2) - (1 + \eta^2 - \sigma^2)\frac{Z_o}{Z}\,A_3\right]\right\} = 0 \tag{6a}$$

$$\left(\frac{\partial}{\partial z} + j\beta_e\right)P_-' + \frac{K}{1+\lambda}\,(A_1 - A_2) = 0 \tag{6b}$$

$$\left(\frac{\partial}{\partial z} + j\beta_e\,\frac{\sigma^2(1-\lambda)}{\eta^2(1+\lambda)N}\right)V' - j\,\frac{\sqrt{\lambda}}{2}\,\frac{\sigma K}{\eta^2(1+\lambda)N}\left[A_1 - A_2 - \frac{(1+\lambda)(1+\eta^2-\sigma^2)}{2\sigma}\,\frac{Z_o}{Z}\,A_3\right] = 0 \tag{6c}$$

$$\left(\frac{\partial}{\partial z} + j\beta_e + j\eta\beta_c\right)Q_+' + K\,A_1 = 0 \tag{6d}$$

$$\left(\frac{\partial}{\partial z} + j\beta_e - j\eta\beta_c\right)Q_-' + K\,A_2 = 0 \tag{6e}$$

$$\left(\frac{\partial}{\partial z} + j\beta_e\right) W' - j\,\frac{\sqrt{\lambda}}{4}\,\frac{1+\eta^2-\sigma^2}{\eta^2+\sigma^2}\,K\,\frac{Z_o}{Z}\,A_3 = 0 \tag{6f}$$

$$\left(\frac{\partial}{\partial z} + j\xi + j\eta\beta_c\right)F_+' - \frac{K}{2}\left[P_+' + P_-' - j\,\frac{4\sigma\sqrt{\lambda}}{1+\lambda}\left(1 + \frac{\eta\beta_c}{\beta_e}N\right)V'\right] = 0 \tag{6g}$$

$$\left(\frac{\partial}{\partial z} + j\xi - j\eta\beta_c\right)F_-' - \frac{K}{2}\left[P_+' - P_-' + j\,\frac{4\sigma\sqrt{\lambda}}{1+\lambda}\left(1 - \frac{\eta\beta_c}{\beta_e}N\right)V'\right] = 0 \tag{6h}$$

$$\left(\frac{\partial}{\partial z} - j\xi + j\eta\beta_c\right)G_+' + \frac{K}{2}\left[P_+' + P_-' - j\,\frac{4\sigma\sqrt{\lambda}}{1+\lambda}\left(1 + \frac{\eta\beta_c}{\beta_e}N\right)V'\right] = 0 \tag{6i}$$

$$\left(\frac{\partial}{\partial z} - j\xi - j\eta\beta_c\right)G_-' + \frac{K}{2}\left[P_+' - P_-' + j\,\frac{4\sigma\sqrt{\lambda}}{1+\lambda}\left(1 - \frac{\eta\beta_c}{\beta_e}N\right)V'\right] = 0 \tag{6j}$$

$$A_1 = \left[J_o(\pi r_o/a) + 2\sum_{n=1}^{\infty} J_{4n}(\pi r_o/a)\cos(4n\psi)\right]\left[\left(1-\sigma\frac{Z_o}{Z}\right)F_+' + \left(1+\sigma\frac{Z_o}{Z}\right)G_+'\right]$$

$$+ 2\sum_{n=1}^{\infty} J_{4n-2}(\pi r_o/a)\cos((4n-2)\psi)e^{-j2\psi}\left[\left(1-\sigma\frac{Z_o}{Z}\right)F_-' + \left(1+\sigma\frac{Z_o}{Z}\right)G_-'\right]$$

$$+ \frac{\eta\beta_c}{\xi}\frac{Z_o}{Z}\frac{\sigma\pi r_o}{a}\sum_{n=1}^{\infty}(-1)^n J_{2n-1}(\pi r_o/a)\Big[(F_+' + G_+')e^{j\psi}e^{j(-1)^n(2n-1)\psi}$$

$$-(F_-' + G_-')e^{-j\psi}e^{-j(-1)^n(2n-1)\psi}\Big] \qquad (7a)$$

$$A_2 = \Big[J_o(\pi r_o/a) + 2\sum_{n=1}^{\infty}J_{4n}(\pi r_o/a)\cos(4n\psi)\Big]\Big[(1-\sigma\frac{Z_o}{Z})F_-' + (1+\sigma\frac{Z_o}{Z})G_-'\Big]$$

$$+ 2\sum_{n=1}^{\infty}J_{4n-2}(\pi r_o/a)\cos((4n-2)\psi)e^{j2\psi}\Big[(1-\sigma\frac{Z_o}{Z})F_+' + (1+\sigma\frac{Z_o}{Z})G_+'\Big]$$

$$+ \frac{\eta\beta_c}{\xi}\frac{Z_o}{Z}\sigma\frac{\pi r_o}{a}\sum_{n=1}^{\infty}(-1)^n J_{2n-1}(\pi r_o/a)\Big[(F_+' + G_+')e^{j\psi}e^{j(-1)^n(2n-1)\psi}$$

$$-(F_-' + G_-')e^{-j\psi}e^{-j(-1)^n(2n-1)\psi}\Big] \qquad (7b)$$

$$A_3 = \Big[J_o(\pi r_o/a) + 2\sum_{n=1}^{\infty}J_{4n}(\pi r_o/a)\cos(4n\psi)\Big]\Big[F_+' - F_-' - G_+' + G_-'\Big]$$

$$-2\sum_{n=1}^{\infty}J_{4n-2}(\pi r_o/a)\cos((4n-2)\psi\;\Big[(F_+' - G_+')e^{j2\psi} - (F_-' - G_-')e^{-j2\psi}\Big] \qquad (7c)$$

$$K = \frac{1}{a}\sqrt{\frac{\pi Z}{2}}\sqrt{\frac{\eta^3}{1+\eta^2-\sigma^2}\;\frac{eI_o}{m\sigma^2 c^2}} \tag{8}$$

In these coupled mode equations, the constants σ, λ, η, and N are as defined in the previous report.[1] In addition, Z_o is the impedance of free space (377 ohms), while ξ and Z are the phase constant and impedance, respectively, for the TE_{10} mode in the square waveguide in the absence of the electron beam.

B. Discussion

Several conclusions can be drawn from the coupled mode equations given above even before any solution is attempted. Since there are ten coupled equations, there should be ten characteristic propagation constants (with, perhaps, a few degeneracies) for any set of parameters. The first step in any attempted solution would be to determine these ten characteristic propagation constants.

If the radius of the electron beam spiral becomes negligible compared to the waveguide dimension, that is, if r_o/a approaches zero, then the ten coupled mode equations reduce essentially to the equations obtained for TEM wave interaction[1]. This occurs because $J_o(\pi r_o/a)$ approaches unity and $J_n(\pi r_o/a)$ approaches zero $(n \geq 1)$ as r_o/a approaches zero. This limiting case is important because the equations become coupled, first order, ordinary differential equations with constant coefficients which may be readily solved. A possible method to solve the equations for finite values of r_o/a, when many of the coefficients are no longer constants but are functions of z through ψ, may be based on this result. The method is to use a perturbation analysis and express the wave amplitudes in terms of series expansions of powers of the parameter r_o/a. For most cases of interest, r_o/a will be small enough that only a few terms will be required in the series to give good results.

As a first step in the solution of the coupled mode equations, then, a discussion of the solutions for $r_o/a = 0$ is necessary. In this case the variation of the field amplitudes over the waveguide cross section has no effect on the solutions, nor does the axial a-c magnetic field have any effect since it is zero on the waveguide axis. To be precise, $r_o/a = 0$ implies a straight filamentary electron beam. However, the viewpoint here will be that the filamentary electron beam spirals, but that the spiral radius is negligible compared to the waveguide dimension as far as the transverse field variation and the axial a-c magnetic field are concerned.

The uncoupled (K = 0) solutions for the ten waves are immediately obtainable from Equations (6a) to (6j) and the ω-β diagrams for these uncoupled waves are shown in Figure 1. The six electron beam waves are shown with solid lines and the four circuit waves with dashed lines.

The coupling between the various waves can have two effects; first, the presence of one of the waves may cause the other waves to be excited, and second, the propagation constants may be perturbed from the uncoupled values. Examination of the coupled mode equations shows that when $r_o/a = 0$, three of the propagation constants for the uncoupled waves are not affected by the presence of finite coupling. These are the propagation constants for the Q_+', Q_-', and W' waves. Thus, three of the ten phase constants sought are

$$\beta = \beta_e + \eta\beta_c \tag{9a}$$

$$\beta = \beta_e - \eta\beta_c \tag{9b}$$

$$\beta = \beta_e \tag{9c}$$

Attention will be directed here primarily to the frequency range in the neighborhood of $\omega = \eta\omega_c$. In this frequency range there is the possibility of strong coupling between the P_+', P_-', F_+', and G_+' waves, with relatively little coupling to the other waves which have uncoupled propagation constants widely separated from

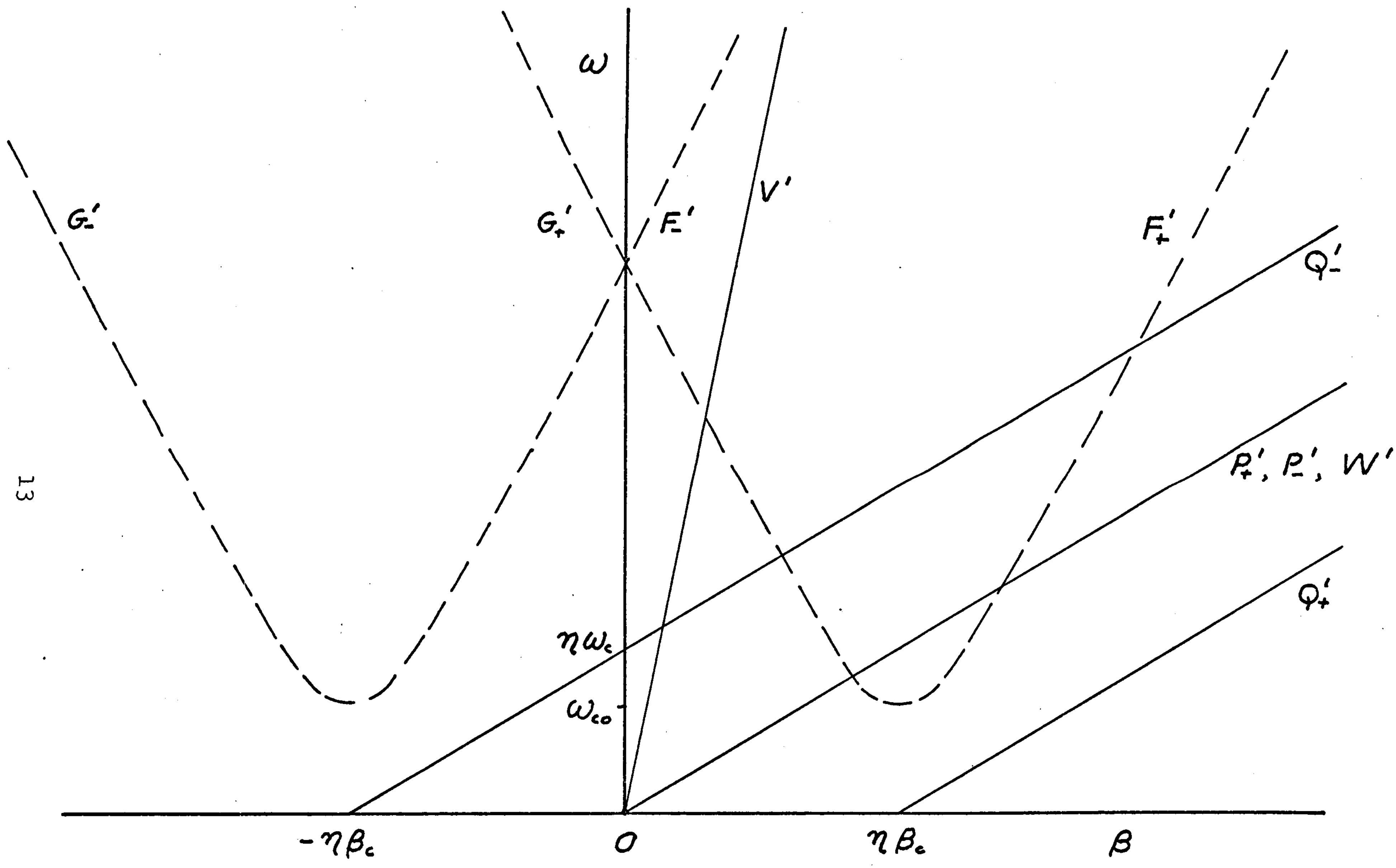

FIGURE 1. The ω − β Diagram for an Uncoupled (K = 0) Spiraling Filamentary Electron Beam in a Square Waveguide. The Six Beam Waves are Shown with Solid Lines, and the Four TE Circuit Waves are Shown with Dashed Lines.

those of the four waves listed. As a consequence, one can take three additional propagation constants as being essentially identical with those of the uncoupled waves, V', F'_-, and G'_- .

$$\beta = \beta_e \ \frac{\sigma^2(1-\lambda)}{\eta^2(1+\lambda)N} \ , \tag{10a}$$

$$\beta = \xi - \eta\beta_c \ , \tag{10b}$$

$$\beta = -\xi - \eta\beta_c \ . \tag{10c}$$

This leaves four propagation constants to be found, with these based on the possibly strong coupling between P'_+, P'_-, F'_+, and G'_+. A typical result for the interaction between these waves is shown in Figure 2. In the frequency neighborhood where the coupling between P'_+, P'_-, and F'_+ is strongest, all four waves have real phase constants. In the frequency neighborhood where the coupling is strongest between P'_+, P'_-, and G'_+, there are two real phase constants and two complex conjugate phase constants.

These results for the very small beam spiral radius (r_o/a approaching zero) can then be used as the zero order solution of the finite beam spiral solution. Examination of the coupled mode equations shows that the perturbation parameter should be $(\pi r_o/a)^2$, since powers of this factor enter into the equations. Note

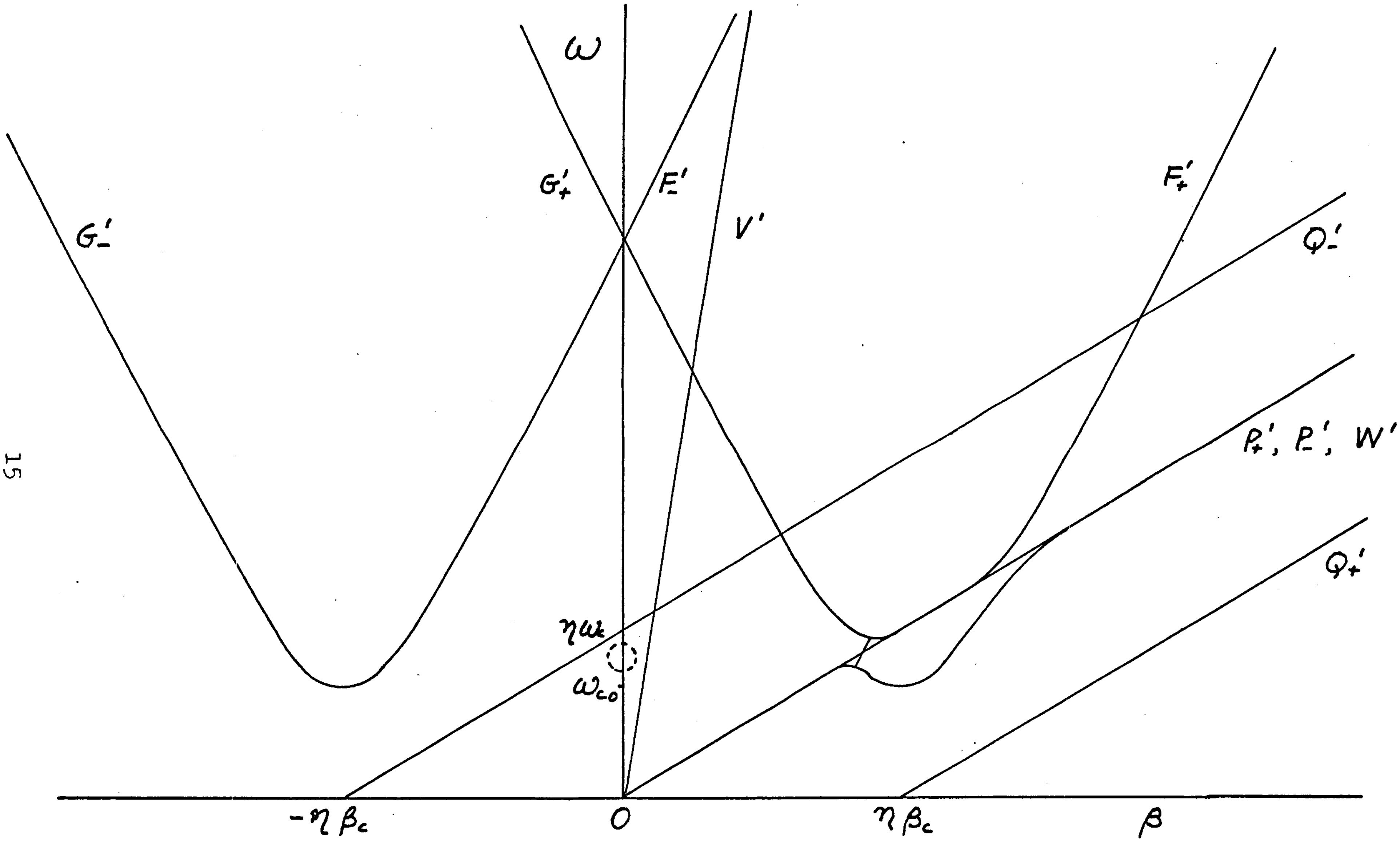

Figure 2. The $\omega - \beta$ Diagram for a Coupled Spiraling Filamentary Electron Beam in a Square Waveguide. The Dotted Lines are the Imaginary Parts of a Complex Conjugate Pair of β Values.

that to be consistent, the various Bessel functions should be expanded in powers of $(\pi r_o/a)$ and only those terms retained which are consistent with the order of the approximation for the solution. The influence of both the transverse field variation and the axial a-c magnetic field will appear in the first order approximation, that is, in terms of order $(\pi r_o/a)^2$. Representative solutions of the finite radius spiraling filamentary electron beam interaction with TE waves will be explored in the next research period.

III. POWER EXCHANGE FOR TRANSVERSE WAVES ON FILAMENTARY ELECTRON BEAMS

A. Introduction

One of the properties of major interest in electron beam-electromagnetic field interaction systems is the power exchanged between the electron beam and the electromagnetic field. In particular, an understanding of the power exchanged is useful in predicting the possibility of amplification or oscillation by the system. When a coupled mode analysis is made, a knowledge of the sign of the power associated with the various modes of the uncoupled elements, together with appropriate power theorems, enables one to predict the character of the possible interactions which can occur.[2]

The sign of the power associated with the modes of an electron beam is a somewhat delicate question. For longitudinal electron beam waves, i.e., space charge waves, there is no ambiguity, and the sign of the power for the fast and slow space charge waves can be determined from a consideration of the beam dynamics. For transverse electron beam waves, however, the sign of the power may depend on the properties of the circuit with which the electron beam interacts.[3] That is, one must specify at least some of the properties of the interaction circuit in order to determine the sign of the power for the transverse electron beam waves.

When a small signal, or linear, analysis of the interaction between an electron beam and an electromagnetic field is made, the various a-c fields, currents, etc., are determined only to first order, and higher order terms are dropped. However, the power, which depends in part on the product of first order a-c quantities, is a second order relation. Therefore, one must exercise care in evaluating the power from the small signal quantities to ensure that all the relevant terms have been properly included.

For example, consider the magnetic force acting on the electrons of the beam. The total force acting on an electron is

$$\overline{F} = -e(\overline{E} + \overline{u}\text{x}\overline{B}) \quad . \tag{11}$$

Clearly there is no net power exchange associated directly with the magnetic field, since

$$\overline{u} \cdot (\overline{u}\text{x}\overline{B}) = 0 \quad . \tag{12}$$

For the moment, assume that the electrons have a d-c velocity only in the z direction and that the magnetic field has a d-c component only in the z direction. In a small signal theory there will be two contributions to a first order, transverse, a-c magnetic force acting on the electron. The first contribution results from the combination of the d-c axial velocity u_{zo} and the a-c transverse magnetic flux density $\overline{B}_{T1}$, while the second results from the combination of the d-c axial magnetic flux density B_{zo} and the a-c transverse velocity $\overline{u}_{T1}$ of the electron. Both of these terms must be included in the calculation of the acceleration of the electron. Of these two transverse magnetic force terms, the first leads to a second order contribution to the power exchange,

$$-e\ u_{xo}\ \overline{u}_{T1} \cdot (\overline{a}_z \times \overline{B}_{T1}) \quad , \qquad (13)$$

while the other term leads to a contribution which is identically zero. The axial magnetic force due to a transverse a-c velocity $\overline{u}_{T1}$ and a transverse a-c magnetic flux density $\overline{B}_{T1}$, given by $-e(\overline{u}_{T1} \times \overline{B}_{T1})$, is a second order term and should not be included in the acceleration expression in a small signal analysis. However, the scalar product of the d-c axial velocity u_{zo} with this a-c magnetic force

$$-e\ u_{zo}\ \overline{a}_z \cdot (\overline{u}_{T1} x \overline{B}_{T1}) \quad , \qquad (14)$$

is a second order term which must be accounted for in the power. In fact, this term will just cancel the previous term so that the net power exchange contains no contribution from the magnetic force, in accordance with (12).

In deriving the power exchanged between the electromagnetic field and the electron beam, usually the equations of motion are used to develop the power expression in a convenient form. The discussion above concerning the magnetic force terms suggests that care must be exercised when the small signal equations of motion of the electron beam are used so that extraneous terms involving the magnetic force are not introduced inadvertently into the expression for the power.

B. Power Exchange for TEM Circuit Wave Interaction

The power absorbed by the electron beam from the electromagnetic field, per unit distance along the beam axis, is

$$\frac{dP}{dz} = \frac{-I_o}{\dot{z}_o} \overline{E} \cdot \overline{u} \quad , \tag{15}$$

where I_o is the d-c current along the beam spiral. In the framework of a small signal theory, the time average power absorbed by the electron beam from the a-c electromagnetic field is

$$\frac{dP}{dz} = \frac{-I_o}{2\dot{z}_o} \mathrm{Re}\,(\overline{E} \cdot \overline{u}_1^*) \quad . \tag{16}$$

The first point to note is that the electric field $\overline{E}$ should be evaluated at the actual position of the electrons in the presence of the a-c electromagnetic field and not at the d-c, or unperturbed, position. That is, the electric field in (16) should be written as

$$\overline{E}(\overline{r}) = \overline{E}(\overline{r}_o + \overline{r}_1) = \overline{E}(\overline{r}_o) + (\overline{r}_1 \cdot \nabla)\overline{E}(\overline{r}_o). \qquad (17)$$

The second term on the right side of (17) is a second order a-c term which is neglected compared to the first order a-c terms in the equations of motion of the small signal analysis. In the expression for the time average power exchange, however, only second order terms contribute, and this one must be included. Thus, correct to second order, the power exchange is

$$\frac{dP}{dz} = \frac{-I_o}{2\dot{z}_o} \text{ Re} \left[\overline{E} \cdot \overline{u}_1{}^* + ((\overline{r}_1{}^* \cdot \nabla)\overline{E}) \cdot \overline{u}_o \right], \qquad (18)$$

where $\overline{E}$ is evaluated at the unperturbed position of the electrons.

For convenience, the discussion will be limited here to the interaction between a spiraling electron beam and the electromagnetic field of a TEM circuit. The method developed can later be applied to the interaction with TE circuits, etc. We assume for the remainder of this report that the electric field $\overline{E}$ and magnetic field $\overline{H}$ are purely transverse, do not vary in the transverse plane, and propagate at the velocity of light in the absence of interaction with the electron beam. For TEM wave interaction, Equation (18) can be written in terms of the circularly polarized components of the fields and velocities as

$$\frac{dP}{dz} = \frac{-I_o}{4\dot{z}_o} \mathrm{Re}\left[E_+' u_+'^* + E_-' u_-'^* - j\,\frac{\eta\zeta}{k}\,(\omega z_1^*)\frac{\partial}{\partial z}(E_+' - E_-') + \frac{\eta^2 \rho \beta_c}{k}\,(\omega z_1^*)(E_+' + E_-')\right] . \tag{19}$$

(The primed quantities were defined previously,[1] $u_\pm' = u_\pm \exp\,(\mp\, j\psi)$, etc.)

The procedure to be followed is to evaluate Equation (19) for the various combinations of beam waves and circuit waves of interest. A necessary prerequisite is to be able to express the electron beam velocity $\overline{u}_1$ and displacement $\overline{r}_1$ in terms of the beam wave eigenvectors. From the definitions of the beam waves,[1]

$$\begin{bmatrix} P_+' \\ P_-' \\ V' \\ Q_+' \\ Q_-' \\ W' \end{bmatrix} = M \begin{bmatrix} 1 & 1 & c_{13} & 0 & 0 & 0 \\ 1 & -1 & c_{23} & 0 & 0 & 0 \\ 0 & 0 & c_{33} & 0 & 0 & 0 \\ 1 & c_{42} & c_{43} & c_{44} & 0 & 0 \\ c_{42} & 1 & -c_{43} & 0 & -c_{44} & 0 \\ c_{61} & -c_{61} & c_{33} & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} u' \\ u' \\ \dot{z}_1 \\ \omega_c r' \\ \omega_c r' \\ \omega z_1 \end{bmatrix} \tag{20}$$

where for the TEM circuit wave interaction,

$$M = \frac{1}{4}\left[(2+\rho^2)\left(\frac{1+\rho^2}{1-\sigma^2}\right)^{1/2}\frac{mI_o}{e}\right]^{1/2}$$

$$c_{13} = j2\,\frac{\eta\beta_c}{\beta_e}\,\frac{\sigma}{(1-\sigma^2)^{3/2}}\,\frac{\rho}{(1+\rho^2)^{1/2}}$$

$$c_{23} = j2\,\frac{\sigma}{(1-\sigma^2)^{1/2}}\,\frac{\rho}{(1+\rho^2)^{1/2}}$$

$$c_{33} = \left[\frac{1+\rho^2}{(1-\sigma^2)(2+\rho^2)}\right]^{1/2}$$

$$c_{42} = \frac{-\rho^2}{2+\rho^2}$$

$$c_{43} = j\,\frac{2\sigma}{(1-\sigma^2)^{1/2}}\,\frac{\rho(1+\rho^2)^{1/2}}{2+\rho^2}$$

$$c_{44} = -j\,\frac{2}{2+\rho^2}\left(\frac{1-\sigma^2}{1+\rho^2}\right)^{1/2}$$

$$c_{61} = -j\,\frac{\sigma}{2}\,\frac{\rho}{(2+\rho^2)^{1/2}}\,\frac{1+\rho^2}{1+\sigma^2\rho^2}$$

$$c_{66} = \frac{-j}{1+\sigma^2\rho^2}\left[\frac{(1-\sigma^2)(1+\rho^2)}{2+\rho^2}\right]^{1/2} \qquad (21)$$

In these, $\rho = \omega_c r_o/c$ and $\sigma = \dot{z}_o/c$. By inverting the matrix,

$$\begin{bmatrix} u' \\ u' \\ \dot{z}_1 \\ \omega_c r' \\ \omega_c r' \\ \omega z_1 \end{bmatrix} = M^{-1} \begin{bmatrix} 1/2 & 1/2 & d_{13} & 0 & 0 & 0 \\ 1/2 & -1/2 & d_{23} & 0 & 0 & 0 \\ 0 & 0 & d_{33} & 0 & 0 & 0 \\ d_{41} & d_{42} & d_{43} & d_{44} & 0 & 0 \\ -d_{41} & d_{42} & d_{53} & 0 & -d_{44} & 0 \\ 0 & d_{62} & d_{63} & 0 & 0 & d_{66} \end{bmatrix} \begin{bmatrix} P' \\ P' \\ V' \\ Q' \\ Q' \\ W' \end{bmatrix} \tag{22}$$

where

$$d_{13} = -(c_{13} + c_{23})/2c_{33}$$

$$d_{23} = -(c_{13} - c_{23})/2c_{33}$$

$$d_{33} = 1/c_{33}$$

$$d_{41} = -(1 + c_{42})/2c_{44}$$

$$d_{42} = -(1 - c_{42})/2c_{44}$$

$$d_{43} = \left[c_{13}(1 + c_{42}) + c_{23}(1 - c_{42}) - 2c_{43}\right]/2c_{33}c_{44}$$

$$d_{44} = 1/c_{44}$$

$$d_{53} = \left[c_{13}(1 + c_{42}) - c_{23}(1 - c_{42}) + 2c_{43}\right]/2c_{33}c_{44}$$

$$d_{62} = -c_{61}/c_{66}$$

$$d_{63} = (c_{23}c_{61} - c_{33})/c_{33}c_{66}$$

$$d_{66} = 1/c_{66} \tag{23}$$

In addition, we find that for TEM circuit waves propagating in the +z direction,

$$E'_{\pm} = 2\sqrt{\frac{Z_o}{A}}\; F'_{\pm} \quad , \tag{24a}$$

while for TEM circuit waves propagating in the -z direction,

$$E'_{\pm} = 2\sqrt{\frac{Z_o}{A}}\; G'_{\pm} \tag{24b}$$

Consider first the interaction between the P'_+ beam wave and the F'_+ circuit wave (both waves propagate in the +z direction). Equation (19) becomes

$$\frac{dP}{dz} = \frac{-I_o}{4M\dot{z}_o}\sqrt{\frac{Z_o}{A}}\;\mathrm{Re}\,(F'_+ P'^*_+) \quad . \tag{25}$$

The circuit wave amplitude F'_+ can be eliminated by using the appropriate equation of motion relating P'_+ and F'_+ (Equation 20a of reference 1),

$$\frac{dP}{dz} = \frac{1}{(2+\rho^2)(1-\sigma)\left[1+\frac{\sigma}{1-\sigma^2}\,\frac{\rho}{1+\rho^2}\,\frac{\eta\beta_c}{\beta_e}\right]}\left[P'^*_+\left(\frac{\partial}{\partial z} + j\beta_e\right)P'_+ + P'_+\left(\frac{\partial}{\partial z} - j\beta_e\right)P'^*_+\right] \quad . \tag{26}$$

This can be simplified and integrated to give the power associated with the P'_+ beam wave,

$$P = \frac{P_+^{'*} \; P_+^{'}}{(2+\rho^2)(1-\sigma)\left[1 + \frac{\sigma}{1-\sigma^2}\,\frac{\rho}{1+\rho^2}\,\frac{\eta\beta_c}{\beta_e}\right]} \,. \tag{27}$$

This power is positive for all d-c beam conditions.

In a similar manner the power associated with the $P_+^{'}$ beam wave when it is excited by the $G_+^{'}$ circuit wave is

$$P = \frac{P_+^{'*} \; P_+^{'}}{(2+\rho^2)(1-\sigma)\left[1 - \frac{\sigma}{1-\sigma^2}\,\frac{\rho}{1+\rho^2}\,\frac{\eta\beta_c}{\beta_e}\right]} \,. \tag{28}$$

The result is slightly different in this case where the circuit wave propagates in the -z direction. However, examination of (28) shows that again the power is positive for all physically realizable d-c beam conditions. Evidently, the $P_+^{'}$ beam wave is a positive energy wave for all TEM circuit wave interactions.

It is not possible to consider the power associated with the $P_-^{'}$ beam wave separately from the $P_+^{'}$ beam wave, because $P_-^{'}$ is not an eigenvector of the beam system.[1] Therefore, both $P_+^{'}$ and $P_-^{'}$ must be considered simultaneously whenever $P_-^{'}$ is excited. Using the procedure outlined above, the power associated with the $P_+^{'}$ and $P_-^{'}$ beam waves when they interact with a $F_+^{'}$ circuit wave is

$$P = \frac{P_+^{'*}\;P_+^{'}}{(2+\rho^2)(1-\sigma)\left[1 + \frac{\sigma}{1-\sigma^2}\,\frac{\rho^2}{1+\rho^2}\,\frac{\eta\beta_c}{\;}\right]} + \frac{(1+\rho^2)(1+\sigma\rho^2)}{(2+\rho^2)(1-\sigma)}\;P_-^{'*}\;P_-^{'}$$

$$+\; j\rho^2\,\eta\beta_c\left[\frac{1}{\left[1 + \frac{\sigma}{1-\sigma^2}\,\frac{\rho^2}{1+\rho^2}\,\frac{\eta\beta_c}{\beta_e}\right]}\right.$$

$$\left. +\; \frac{K^2\sqrt{1+\rho^2}}{4\beta_e^2\sqrt{1-\sigma^2}}\right]\int_0^z (P_+^{'*}\;P_-^{'} - P_+^{'}\;P_-^{'*})\,dz, \qquad (29)$$

and when they interact with a $G_+^{'}$ circuit wave is

$$P = \frac{P_+^{'*}\;P_+^{'}}{(2+\rho^2)(1+\sigma)\left[1 - \frac{\sigma}{1-\sigma^2}\,\frac{\rho^2}{1+\rho^2}\,\frac{\eta\beta_c}{\beta_e}\right]} + \frac{(1+\rho^2)(1-\sigma\rho^2)}{(2+\rho^2)(1+\sigma)}P_-^{'*}P_-^{'}$$

$$+\; j\rho^2\eta\beta_c\left[\frac{1}{\left[1 - \frac{\sigma}{1-\sigma^2}\,\frac{\rho^2}{1+\rho^2}\,\frac{\eta\beta_c}{\beta_e}\right]}\right.$$

$$\left. -\; \frac{K^2\sqrt{1+\rho^2}}{4\beta_e^2\sqrt{1-\sigma^2}}\right]\int_0^z (P_+^{'*}P_-^{'} - P_-^{'}P_+^{'*})\,dz \quad . \qquad (30)$$

In each case the first two terms in the expression for the power are positive. The sign of the third term in each case depends on the relative phase of $P_+^{'}$ and $P_-^{'}$.

It appears that further study involving particular electron beam and circuit structures with their associated boundary conditions is necessary to explore this power and its possible sign.

This same procedure can be applied to the other beam waves for their possible interactions with the circuit waves in order to determine their associated power. Examination of Equations (19) and (22) shows that the Q_+' and Q_-' beam waves will not couple to a TEM circuit wave. The V' beam wave can couple to either the G_+' or the F_-' circuit waves. The power associated with the V' beam wave when it is excited by the G_+' circuit wave is

$$P = \frac{4V'^* V'}{(1+\sigma)\sqrt{2+\rho^2}} \left[\frac{\eta}{1+\sigma^2\rho^2} + \frac{\sigma\sqrt{2+\rho^2}}{1+\rho^2} \left(1 + \frac{\eta\beta_c/\beta_e}{1-\sigma^2}\right)\right], \tag{31}$$

and when it is excited by the F_-' circuit wave is

$$P = \frac{4V'^* V'}{(1+\sigma)\sqrt{2+\rho^2}} \left[\frac{\eta}{1+\sigma^2\rho^2} - \frac{\sigma\sqrt{2+\rho^2}}{1+\rho^2} \left(1 - \frac{\eta\beta_c/\beta_e}{1-\sigma^2}\right)\right]. \tag{32}$$

In the first case the power is always positive; in the second case the power may be negative, but only for extremely high d-c beam velocities. The W' beam wave can interact with either the F_+' or the G_+' circuit waves. In each case the power is

$$P = - \frac{4(1 + \sigma^2 \mathcal{S}^2)^2 \, W'^* \, W'}{(1-\sigma^2)(1+\mathcal{S}^2)} \tag{33}$$

This is negative for all d-c beam conditions.

The implications of these results for the power associated with the beam waves when interacting with various TEM circuit waves must be explored further to clarify the possibilities for amplification and oscillation of the electron beam - circuit system. This general procedure can also be used to explore the power associated with the beam waves when they are excited by TE or TM circuit waves.

References

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